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Planar Finite Automata

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Abstract

Several aspects of the graph representations of finite automata have been studied in the literature, e.g., the complexity considerations involving the number of states. It is natural to ask, whether we indeed need the full generality of possible interconnections among the states provided by the general definition of finite automata. In the present paper we shall study the influence of restricting the interconection network among the states to planar graphs.

We introduce planar finite automata as a type of finite automata whose graph representation is planar. We study families of languages defined by deterministic, nondeterministic and ε-free nondeterministic planar finite automata and compare them to $\mathcal{R}$, the family of regular languages. Deterministic planar finite automata are shown to be weaker than finite automata, except for the case of a one-letter alphabet. The equality of $\mathcal{R}$ with the family of languages accepted by (ε-free) nondeterministic planar automata is proven.

We also study the influence of restricting the interconnections among the states of the finite automaton to a particular architecture studied in parallel and distributed computing, namely the d dimendional mesh. We study an extension of $k\varepsilon$-bounded nondeterministic finite automata with the maximal number of consecutive ε moves bounded by k. We embed the graph representation of such automata into a mesh and show that for any k there is a language which cannot be accepted by $k\varepsilon$-bounded nondeterministic finite automaton embedded into the mesh.

1 Introduction and Definitions

A finite automaton is the simplest model of computation having only a finite amount of memory. It operates on sequences of input symbols — input words. It has states, and depending on the input passes from one state to another in discrete time steps. The knowledge of its current state is its sole way of storing information.

Formally, a *finite automaton* A is a 5-tuple $(K, \Sigma, \delta, q_0, F)$, where K is the set of all states, Σ is the input alphabet, $\delta : K \times (\Sigma \cup \{\varepsilon\}) \to 2^K$ is the transition function specifying the moves from one state to another, $q_0 \in K$ is the initial state and $F \subseteq K$ is the set of final states. A word $w = a_1 a_2 ... a_s \in \Sigma^*$ is *accepted* by A, if A passes through states $q_0, q_1, ..., q_s$ reading the input word w. This means that $q_i \in \delta(q_{i-1}, a_i)$ for $i = 1, 2, ..., s$ and $q_s \in F$. We call it a *computation* on w and write $(q_0, a_1 a_2 ... a_s) \vdash_A (q_1, a_2 ... a_s) \vdash_A \ldots \vdash_A (q_s, \varepsilon)$.

If for all $a \in \Sigma$ and all $q \in K$

(i) $|\delta(q, a)| = 1$ and

(ii) $|\delta(q, \varepsilon)| = 0$,

the finite automaton A is called *deterministic*, otherwise it is called *nondeterministic*. If only the second property holds, we call it ε*-free*.

All words $w \in \Sigma^*$ accepted by the finite automaton A form a *language* $L(A) \subseteq \Sigma^*$. A language is said to be finite, if it consists of a finite number of words. A family $\mathcal{R}$ of languages accepted by nondeterministic finite automata is called *regular*.

Finite automata have been widely studied. Now we recall some important results (mostly obtained from [2]). The families accepted by nondeterministic, deterministic and ε-free nondeterministic finite automata are equal. It means we can simulate one by another. We have also two very strong characterizations of regular languages, Kleene's (see [3]) and Myhill-Nerode's Theorem (see [4, 5]):

Theorem 1.1 (Kleene) *The family of regular languages over alphabet Σ is the smallest family of subsets of Σ^* which contains the finite languages and is closed under the operations of union $\cup$, concatenation . and star *.*

The *index* of an equivalence relation is the number (finite or infinite) of equivalence classes generated by the relation. The equivalence relation $\equiv$ on the language Σ^* is *right invariant*, if the following holds: if $u \equiv v$, then $ux \equiv vx$ for any $x \in \Sigma^*$.

Theorem 1.2 (Myhill-Nerode) *For each language L over Σ the following three statements are equivalent:*

(i) L is regular.

(ii) There exists a right invariant equivalence relation $\equiv$ on Σ^*, of finite index, such that L is the union of some equivalence classes determined by $\equiv$.

(iii) Let $\equiv_L$ be the relation on Σ^* defined by $u \equiv v$ if for all words $x \in \Sigma^*$, ux is in L if and only if vx is in L. Then $\equiv_L$ is the equivalence relation of finite index.

We have also a very nice characterization of regular languages over one-letter alphabet. The following theorem is a corollary either of Kleene's or Myhill-Nerode's Theorem (see [2, page 48]).

Theorem 1.3 *Let L be a language over $\Sigma = \{a\}$. Then L is regular if and only if there exist numbers p and q and finite sets $S, T \subseteq \mathcal{N}$, $\max_{x \in S} x < r$, $\max_{x \in T} x < p$, so that $L = \{a^i \mid i \in S \text{ or } i \geq r \wedge (i - r) \bmod p \in T\}$.*

A finite automaton $A = (K, \Sigma, \delta, q_0, F)$ can be represented also by its arcs. An *arc* is a triple (p, a, q), where $p, q \in K$ are states, $a \in \Sigma$ is a letter such that $q \in \delta(p, a)$. We can construct a labelled graph $G(A) = (V, E)$ for the automaton

A as follows: The set of vertices V equals to K. The set of edges of G consists of all unordered pairs $\{p, q\}$, such that there is an arc from p to q or from q to p

$$E = \{\{p, q\} \mid \exists a \in \Sigma \cup \{\varepsilon\} \ : \ q \in \delta(p, a)\}$$

We allow loops (edges of form $\{p, p\} = \{p\}$). Next, we label each edge with the set of all arcs *passing* through that edge:

$$\text{label}(\{p, q\}) = \{(p, a, q) \mid a \in \Sigma \cup \{\varepsilon\} \ \wedge \ q \in \delta(p, a) \vee p \in \delta(q, a)\}$$

We put vertices corresponding to final states in double circles in $G(A)$. The initial state is marked by $\rightarrow$.

The graph $G(A)$ is said to be planar, if it can be embedded into a plane.

Definition **1.4** A (deterministic, nondeterministic, ε-free nondeterministic) finite automaton A is said to be *planar*, if its graph $G(A)$ is planar. We abbreviate term planar finite automaton to PFA.

Sometimes we will not discuss loops, as they cannot disturb planarity.

Note that for each planar labeled graph G with given initial vertex and some final vertex (vertices), one can easily construct a planar finite automaton A such that $G(A) = G$.

We recall some properties of planar graphs to be used in the sequel.

Theorem 1.5 (Euler) *Let G be a planar graph with n vertices, m edges and f faces. The following equation holds: $n - m + f = 2$.*

Corollary 1.6 *Let G be a planar graph without loops with n vertices and m edges. Then*

- $m \leq 3n - 6$,
- *if G is bipartite, $m \leq 2n - 4$.*

We recall also the relationship between the number of edges and the degrees of vertices. Let G be a graph with n vertices v_i, $i = 1, 2, ..., n$, and m edges. Let d_i be the degree of vertex v_i. Then $2m = \sum_{i=1}^{n} d_i$. The previous corollary implies that a graph with the average degree $d \geq 6$ or a bipartite graph with $d \geq 4$ cannot be planar.

2 Deterministic Planar Finite Automata

The family of languages accepted by deterministic planar finite automata over the alphabet Σ is shown to be the proper subset of regular languages over Σ. If $|\Sigma|$ is large, the proof is simple. It suffices to find any regular language L which cannot be accepted by a deterministic planar automaton.

Definition **2.1** Let $L_{n,k}$ be a regular language over the alphabet $\Sigma = \{0, 1, ..., n-1\}$ defined as follows:

$$L_{n,k} = \{a_1 a_2 \dots a_p \mid p \geq 0,\ a_i \in \Sigma \ \ \forall i \in \{1, 2, ..., p\}, \quad \sum_{i=1}^{p} a_i \equiv k (\text{mod } n)\}$$

Let $L = L_{n,0}$, $n \geq 6$, and let A be any deterministic finite automaton such that $L(A) = L$. The automaton is deterministic, therefore there are exactly n arcs from every state q in A. It easy follows from the definition of L that no two arcs can lead to the same state (and no arc can return back to q), so there must be at least 6 edges incident to q in $G(A)$ (there may be some arcs leading to q creating some more edges). Thus $G(A)$ has no loops and its average degree is at least 6, which contradicts the planarity of $G(A)$ (see Corollary 1.6).

It is natural to ask what is the situation for alphabets of cardinality less than 6. We shall show, that all regular languages over a single letter alphabet can be accepted by a planar finite automaton.

Theorem 2.2 *Let Σ be an alphabet, $\Sigma = \{a\}$. A family of languages over Σ defined by deterministic planar finite automata is equal to the family of regular languages over Σ.*

Proof: Let $\mathcal{L}_{1DPF}$ be a family of languages over Σ defined by a deterministic PFA. Clearly $\mathcal{L}_{1DPF} \subseteq \mathcal{R}$, since every deterministic PFA is already a finite automaton.

Let L be any regular language in Σ^*. We shall show that there is a deterministic planar automaton A accepting L. By Theorem 1.3 there exist numbers p and q and finite sets $S, T \subseteq \mathcal{N}$, $\max_{x \in S} x < r$, $\max_{x \in T} x < p$, so that $L = \{a^i \mid i \in S \ \text{ or } \ i \geq r \wedge (i - r) \bmod p \in T\}$.

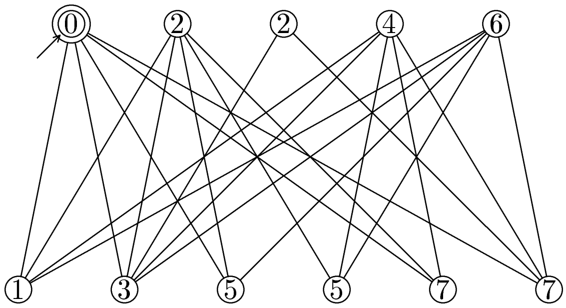
The construction of $A = (K, \Sigma, \delta, q_0, F)$ is done in a straightforward way. Let $K = \{q_0, q_1, ..., q_{r+p-1}\}$. Define the δ-function as follows:

$$\begin{aligned} \delta(q_i, a) &= \{q_{i+1}\} \quad \text{if } 0 \leq i < r + p - 1 \\ \delta(q_{r+p-1}, a) &= \{q_r\} \end{aligned}$$

The set of final states is $F = \{q_i \mid i \in S \ \text{ or } \ i \geq r \wedge (i - r) \bmod p \in T\}$.

The graph $G(A)$ consists of a circle and a path and is evidently planar. It proves $\mathcal{R}_{\{a\}} \subseteq \mathcal{L}_{1DPF}$. $\square$

The following theorem shows an example of a regular language over a two-letter alphabet which cannot be accepted by a deterministic planar finite automaton. Therefore for any Σ, $|\Sigma| \geq 2$, the family of languages over Σ accepted by deterministic planar finite automata is a proper subset of the family of regular languages over Σ.



Graph of an automaton accepting the language
L from the Theorem 2.3 cannot be planar.

Theorem 2.3 *There exists a regular language over alphabet Σ, $|\Sigma| = 2$ which cannot be accepted by a deterministic planar finite automaton.*

Proof: Let $\Sigma = \{1, 5\}$ and

$$L = \{a_1 a_2 ... a_k \mid k \in \mathcal{N},\ \text{each } a_i \in \Sigma,\ \sum_{i=1}^{k} a_i \equiv 0 \pmod 8\}.$$

Suppose there is a deterministic PFA A accepting L. By the Myhill-Nerode's Theorem (1.2) there are at least 8 classes of equivalence of the states of A, each corresponding to a class of congruence modulo 8. Therefore, from any state q corresponding to the ith congruence class there has to be an arc to some state p from $((i+1) \bmod 8)$th class and to some state r from $((i+5) \bmod 8)$th class. Of course, none of p, q and r correspond to the same class. Furthermore, if there is an arc from q to p, there cannot be an arc from p to q. This implies that for every single arc there is its corresponding edge.

We can orientate to every edge according to orientation of the arc passing through it. In such an oriented graph, the outdegree of every vertex is exactly $|\Sigma| = 2$. Therefore the average of indegrees is also 2. This implies, that the average degree of $G(A)$ is 4. Furthermore, it is bipartite, since from the "even" class there are only edges to "odd" classes and vice versa. By Corollary 1.6 $G(A)$ can not be planar. $\square$

Corollary 2.4 *A family of languages defined by deterministic planar finite automata is a proper subset of $\mathcal{R}$.*

3 Nondeterministic Planar Finite Automata

In this section we prove that a nondeterministic planar finite automaton with only one final state is a normal form of finite automata. Furthermore, we claim that the initial and final states can be placed on the boundary of the outside face of its graph.

Definition **3.1** Let $A = (K, \Sigma, \delta, q_0, F)$ be a nondeterministic PFA. We call it *closed* nondeterministic PFA, if $|F| = 1$ and the graph $G(A)$ can be embedded into the plane so that vertices corresponding to q_0 and $f \in F$ are on the boundary of its outside face. This embedding of graph $G(A)$ is said to be *closed*.

Lemma 3.2 *Let $A_i = (K_i, \Sigma_i, \delta_i, q_{0i}, \{f_i\})$, $i = 1, 2$, be two closed nondeterministic PFA. Let $L_i = L(A_i)$, $i = 1, 2$. Then there exists a closed nondeterministic PFA A for language L, where*

a) $L = L_1 \cup L_2$,
b) $L = L_1 L_2$,
c) $L = L_1^*$.

Proof: Suppose $K_1 \cap K_2 = \emptyset$ (if not, rename the states). We shall construct a closed nondeterministic PFA $A = (K, \Sigma, \delta, q_0, F)$ by constructing the graph $G(A)$ from the graphs $G(A_i)$ by placing them into the plane so that each is in the outside face of the other and by adding new ε-arcs.

a) We simply add an ε-arc from q_{01} to q_{02} and from f_2 to f_1. The initial and the final states of A are the same as those of A_1. We thus obtain a closed embedding of $G(A)$.

   Formally, $A = (K_1 \cup K_2, \Sigma_1 \cup \Sigma_2, \delta, q_{01}, \{f_1\})$, where
$$\begin{aligned} \delta(q,z) &= \delta_1(q,z), \quad \text{if } q \in K_1 - \{q_{01}\} \text{ and } z \in \Sigma_1 \cup \{\varepsilon\}, \\ \delta(q,z) &= \delta_2(q,z), \quad \text{if } q \in K_2 - \{f_2\} \text{ and } z \in \Sigma_2 \cup \{\varepsilon\}, \\ \delta(q_{01},\varepsilon) &= \delta_1(q_{01},\varepsilon) \cup \{q_{02}\} \\ \delta(f_2,\varepsilon) &= \delta_2(f_2,\varepsilon) \cup \{f_1\} \end{aligned}$$

b) We add an ε-arc from f_1 to q_{02}. The initial state of A is q_{01} and the final state of A is f_2. The graph $G(A)$ so constructed is closed embedded into the plane.

   Formally, $A = (K_1 \cup K_2, \Sigma_1 \cup \Sigma_2, \delta, q_{01}, \{f_2\})$, where
$$\begin{aligned} \delta(q,z) &= \delta_1(q,z), \quad \text{if } q \in K_1 - \{f_1\} \text{ and } z \in \Sigma_1 \cup \{\varepsilon\}, \\ \delta(q,z) &= \delta_2(q,z), \quad \text{if } q \in K_2 \text{ and } z \in \Sigma_2 \cup \{\varepsilon\}, \\ \delta(f_1,\varepsilon) &= \delta_1(f_1,\varepsilon) \cup \{q_{02}\} \end{aligned}$$

c) We add an ε-arc from f_1 to q_{01}. Both the initial and the final state of A is q_{01} (because $\varepsilon \in L_1^*$).

Formally, $A = (K_1, \Sigma_1, \delta, q_{01}, \{q_{01}\})$, where

$$\begin{array}{lcl} \delta(q, z) & = & \delta_1(q, z), \quad \text{if } q \in K_1 - \{f_1\} \text{ and } z \in \Sigma_1 \cup \{\varepsilon\}, \\ \delta(f_1, \varepsilon) & = & \delta_1(f_1, \varepsilon) \cup \{q_{01}\} \end{array}$$

In all three cases, the proof of the identity $L(A) = L$ is straightforward. □

We use the previous lemma for proving that closed nondeterministic planar finite automata are a normal form of finite automata.

Theorem 3.3 *For any regular language L there exists a closed nondeterministic planar finite automaton A_P such that $L = L(A_P)$.*

Proof: By the Kleene's Theorem 1.1, we can construct the regular language L from finite languages using a finite number of operations $\cup$, . and *.

We prove the theorem by induction on the number of operations in the construction.

Let $L = \{w\}$, $w = a_1 a_2 ... a_n \in \Sigma^*$ be a finite regular language over Σ. Then $G(A_P)$ is a path and therefore A_P is a closed nondeterministic PFA for L. Formally $A_P = (K, \Sigma, \delta, q_0, \{q_{|w|}\})$, where $K = \{q_0, q_1, ..., q_{|w|}\}$ and $\delta(q_{i-1}, a_i) = \{q_i\}$, $i = 1, 2, ..., n$.

Finite languages consisting of more words can be constructed from one-word languages and the union operation and therefore are not discussed here particularly.

If $L = L_1 \cup L_2$, or $L = L_1 L_2$, or $L = L_1^*$, by inductive hypothesis there is a closed nondeterministic PFA A_i such that $L(A_i) = L_i$, $i = 1(, 2)$. The assumptions of Lemma 3.2 are fullfilled and the existence of a closed nondeterministic PFA A_P follows. □

The following corollary is a direct consequence of Theorem 3.3.

Corollary 3.4 *The family of languages defined by nondeterministic planar finite automata is equal to $\mathcal{R}$.*

Next we shall prove the equality of the family of regular languages $\mathcal{R}$ and the family of languages defined by ε-free nondeterministic planar finite automata. First we show a construction of an ε-free nondeterministic PFA for a special type of regular languages. It illustrates the idea used in the proof of the general statement in Theorem 3.6.

Example 3.5 *We shall construct an ε-free nondeterministic PFA $A_{n,k}$ accepting $L_{n,k}$, $n > k \geq 0$ (see Definition 2.1).*

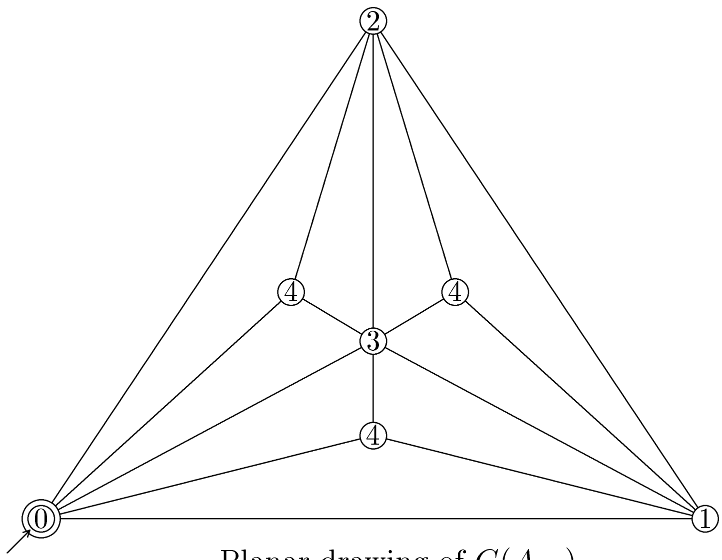
We exploit nondeterminism to a large extent. First, we show the construction for a language $L_{n,0}$. Let A be a deterministic finite automaton accepting $L_{n,0}$, having exactly n states, each corresponding to the class of congruence modulo n. More precisely, $A = (\{p_0, p_1, ..., p_{n-1}\}, \{0, 1, ..., n-1\}, \delta, p_0, \{p_0\})$, where $\delta(p_i, a) = \{p_{(i+a) \bmod n}\}$ (the automaton is in the state p_i after reading the input word $a_1 a_2 ... a_m$, where $\sum_{j=1}^{m} a_j \equiv i \pmod{n}$).

We shall construct $A_{n,0}$ by making appropriate number of copies of each state p_i. We denote these states by $q_{i,c}$, $c = 1, 2, ..., n_i$, where n_i is the number of the copies.

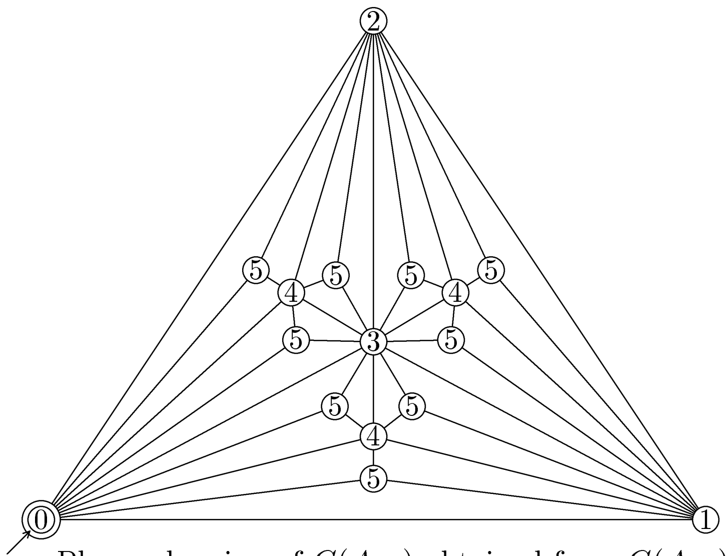
Suppose we are given an ε-free nondeterministic PFA $A_{n-1,0}$ and its graph $G(A_{n-1,0})$. We shall construct $A_{n,0}$ together with $G(A_{n-1,0})$ by using all the states of $A_{n-1,0}$ and adding several new states corresponding to the residue $(n-1)$ modulo n. We have to deal with a problem that the transition function works according to modulus $(n-1)$ in $A_{n-1,0}$. For example when reading 1 the automaton $A_{n-1,0}$ moves from the state q_{n-2} to the state q_0. Since this does not work for $A_{n,0}$, we relabel the arcs of $G(A_{n-1,0})$ according to modulus n in order to create $G(A_{n,0})$. We embed it into a plane and add several new vertices, one to each face (except the outside face). We denote them by $q_{n-1,c}$, $c = 1, 2, ..., f-1$, where f is the number of faces.

We place an edge from $q_{n-1,c}$ to $q_{m,d}$ if and only if $q_{n-1,c}$ lies in a face, that has $q_{m,d}$ on its boundary. We add arcs $(q_{n-1,c}, m-(n-1) \pmod{n}, q_{m,d})$ and $(q_{m,d}, (n-1)-m, q_{n-1,c})$. We thus obtain a graph of $A_{n,0}$, where $q_{0,c}$ is both the initial and the final state (there is only one c such that $q_{0,c}$ is a state, because it is an initial state of $A_{n-1,0}$).

From $A_{n,0}$ constructed above we easily derive $A_{n,k}$. They have associated the same graph and they differ only in the final states; in $A_{n,k}$, the set of final states is $\{q_{k,c} \mid 1 \leq c \leq n_k\}$.



Planar drawing of $G(A_{5,0})$



Planar drawing of $G(A_{6,0})$ obtained from $G(A_{5,0})$

Theorem 3.6 *For any $L \in \mathcal{R}$ there exists an ε-free nondeterministic planar finite automaton A_P accepting L.*

Proof: Let $A = (K, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton accepting L. Let $K = \{q_0, q_1, ..., q_n\}$. Further, let A_i be a deterministic finite automaton derived from A by taking only states $K_i = \{q_0, q_1, ..., q_i\}$. It means that $A_i = (K_i, \Sigma, \delta_i, q_0, F_i)$, where $\delta_i = \delta/_{K_i \times \Sigma}$ and $F_i = F \cap K_i$.

We construct A_P by induction on i. We construct for each A_i an ε-free nondeterministic PFA A_{Pi} such that $L(A_i) = L(A_{Pi})$. Then $A_P = A_{Pn}$.

The idea behind is very similar to the idea used in the previous example. We copy every state q_j of A_i as many times as needed. First we define a *coloured* planar graph G'_i for A_{Pi}, with the edges coloured by red and green, where red means "STOP" and green means "GO". From this graph we derive graph G_i for A_{Pi} by taking the green edges only.

The set of vertices of G'_i equals to the set of states of A_{Pi} which consists of several copies of the states from K_i. We denote each copy of the state q_j by (q_j, c), $j = 0, 1, ..., i$, $c = 1, 2, ..., n_j$, where n_j is the number of copies of q_j. If there is an edge from p_1 to p_2 in G'_i, it is red if and only if there is no possible move according to the δ_i-function from the state q_1 to the state q_2, where $p_k = (q_k, c_k)$ is one of the copies of q_k, $k = 1, 2$. Otherwise (when $q_2 \in \delta_i(q_1, a)$ or $q_1 \in \delta_i(q_2, a)$ for some $a \in \Sigma$) the edge $\{p_1, p_2\}$ is green. Automaton A_{Pi} is obtained this way: There is an arc (p_1, a, p_2) in A_{Pi} whenever $p_2 \in \delta_i(p_1, a)$ and there is an edge $\{p_1, p_2\}$ in G'_i for every $a \in \Sigma$ and vertices p_1, p_2. The initial state is $(q_0, 1)$, the final states are all states of form (q_F, c), where q_F is a final state of A_i. Clearly, $L(A_{Pi})$ is a subset of $L(A_i)$. We shall show the second inclusion, $L(A_i) \subseteq L(A_{Pi})$.

We prove by induction on i that there exists a coloured planar graph G'_i. Further we show that for every sequence $r_1, r_2, ..., r_s$ ($r_k \in K_i$ for $k = 1, 2, ..., s$) there exists a path [1] $p_1, p_2, ..., p_s$ in G'_i such that $p_k = (r_k, c_k)$, $k = 1, 2, ..., s$. We call the path a *representation* of $r_1, r_2, ..., r_s$. It does not matter what are the colours of the edges in the representation.

For $i \in \{0, 1\}$, we can easily embed the graph $G(A_i)$ into the plane. Therefore $A_{Pi} = A_i$ would hold in this case. To unify the presentation for all i, we denote the states of $A_{Pi} = (K_{Pi}, \Sigma, \delta_{Pi}, (q_0, 1), F_{Pi})$ for $i \in \{0, 1\}$ as follows: $K_{Pi} = K_i \times \{1\}$, $F_{Pi} = F_i \times \{1\}$, thus $\delta_{Pi}((q, 1), a) = \{(p, 1) \mid \delta_i(q, a) = p\}$. We construct G'_i from $G(A_i)$ as follows: We add a loop $\{(q, 1), (q, 1)\}$ for each $q \in K_i$. If there is an $a \in \Sigma$ such that $\delta_i(q, a) = q$, $q \in K_i$, we colour the loop by green, otherwise by red. For $i = 1$ we add edge $\{(q_0, 1), (q_1, 1)\}$ into G'_1 and colour it according to the δ_1-function.

Let $i > 1$. By the inductive hypothesis there exists a coloured planar graph G'_{i-1}. We construct the coloured planar graph G'_i from the given G'_{i-1} as follows. Let e_i be the number of edges of G'_{i-1} (the colour does not matter). We simply copy e_i-times the state q_i and create new states (q_i, c), where $c = 1, 2, ..., e_i$. Then we place one of these states (vertices) near every edge (which is not a loop) of G'_{i-1} (we place it to one of its adjacent faces). Let (p_1, p_2) be one of the edges and p_3 be the new vertex corresponding to it. We join by an edge both of these pairs of vertices: $\{p_1, p_3\}$, $\{p_2, p_3\}$. Next, we place a loop into G'_i from every (q_i, c) to itself. Each new edge is coloured green, if there is an arc in A_{Pi} passing through

[1] By term path we mean any sequence of (not necessarily distinct) vertices, such that every two consecutive vertices share an edge.

it, otherwise it is coloured red. We define the initial state as $(q_0, 1)$ and the final states as $F_{Pi} = \{(q_k, c) \mid q_k \in F_i, 1 \le c \le n_k\}$. This completes the definition of G'_i.

Clearly none of the newly introduced edges violate the planarity and therefore the constructed graph G'_i is planar. Now we prove the second part. Let $r_1, r_2, ..., r_s$, $r_k \in K_i$ for $k = 1, 2, ..., s$, be any sequence of states. Let $r_{i_1}, r_{i_2}, ..., r_{i_m}$ be the subsequence obtained from $r_1, r_2, ..., r_s$ by omitting all elements q_i. By the inductive hypothesis there exists a representation $v_1, v_2, ..., v_m$ of $r_{i_1}, r_{i_2}, ..., r_{i_m}$ in G'_{i-1}. By induction on $s - m$ one can easily show, that there is a representation of $r_1, r_2, ..., r_s$ in G'_i.

We prove now $L(A_i) \subseteq L(A_{Pi})$. Let $w = a_1 a_2 ... a_s \in L(A_i)$ and $(r_0, a_1 a_2 ... a_s) \vdash (r_1, a_2 ... a_s) \vdash \ldots \vdash (r_s, \varepsilon)$, where $r_k \in K_i$, $k = 0, 1, ..., s$, and $r_s \in F_i$, be the computation of A_i on w. In the graph G'_i there exists a representation $p_0, p_1, ..., p_s$ of $r_0, r_1, ..., r_s$. Since for every $k = 1, 2, ..., s$ it holds $(r_{k-1}, a_k w_{k+1}) \vdash (r_k, w_{k+1})$, $w_{k+1} = a_{k+1} ... a_s$, the edge from p_{k-1} to p_k has to be coloured by green (and there is an arc $(p_{k-1}, a + k, p_k)$). Therefore the representation $p_0, p_1, ..., p_s$ is also a path in G_i and w is accepted by A_{Pi}.

Now it suffices to prove the inclusion $L(A_{Pi}) \subseteq L(A_i)$. Let $w = a_1 a_2 ... a_s \in L(A_{Pi})$ and $(p_0, a_1 a_2 ... a_s) \vdash (p_1, a_2 ... a_s) \vdash \ldots \vdash (p_s, \varepsilon)$, where $p_k \in K_{Pi}$, $k = 0, 1, ..., s$, and $p_s \in F_{Pi}$, be the computation of A_{Pi} on w. Let the kth step of the computation on w be $(a_k w_{k+1}, p_{k-1}) \vdash (w_{k+1}, p_k)$, where $p_{k-1} = (q_l, c)$ and $p_k = (q_m, d)$. Since the graph G_i was constructed so that $(a_k w_{k+1}, q_l) \vdash (w_{k+1}, q_m)$ in A_i, there exists a computation of A_i on w. $\square$

In the previous proof we copied the state q_i too many times to make the proof more understandable. It is sufficient to copy it into every face of G'_{i-1} (and we can exclude the outside face). From there we know that the number of copies of the state q_i, $i \ge 3$, can be bounded by 3^{i-3}. It means that we are able to construct an ε-free nondeterministic planar finite automaton accepting the language $L \in \mathcal{R}$ with at most $3 + \sum_{i=3}^{n} 3^{i-3} = 3 + \frac{3^{n-2}-1}{2}$ states. This upper bound is exponential in n, where n is index of the equivalence relation of the language L defined by the Myhill-Nerode's Theorem 1.2.

Corollary 3.7 *The family of languages defined by ε-free nondeterministic planar finite automata equals to $\mathcal{R}$.*

4 Embedding into a mesh

The model discussed in the previous section has an unpleasant property: the number of vertices reachable from a given vertex in n steps can be exponential in the length n of a path. There were also no restrictions on the degree of the vertices. Further, the lengths of edges were not bounded. In this section, we discuss a more realistic model, where the graph of an automaton must be embedded in a mesh. We allow a bounded number of ε moves to obtain a less restrictive model.

Definition **4.1** Let A be a nondetermistic finite automaton. We say that A is *$k\varepsilon$-bounded*, if for any word w and for any computation A on w it holds that A never uses more than k consecutive ε moves.

Recall that a graph G can be *embedded* into graph H, if there is a subgraph of H isomorphic to G.

Definition **4.2** A finite automaton A is an *d-mesh* finite automaton, if its graph $G(A)$ can be embedded into a d-dimensional mesh.

In the case of one letter alphabet, for every language L we can construct a 2-mesh deterministic (i.e. also 0ε-bounded) automaton accepting L (construction is similar to the proof of the Theorem 2.2, we can assume that p is even).

Theorem 4.3 *For each $d \in \mathcal{N}$ and each $k \in \mathcal{N}$ there is a regular language over two letter alphabet which cannot be accepted by a $k\varepsilon$-bounded d-mesh finite automaton.*

Proof: Let $\mathcal{L}_n = \{L \mid L \subseteq \{a,b\}^n\}$ be the family of all languages over the alphabet $\{a,b\}$ consisting of n letter words. We shall show that for sufficiently large n, there exists $L \in \mathcal{L}_n$ which cannot be accepted by a $k\varepsilon$-bounded d-mesh finite automaton.

Let L be a language from $\mathcal{L}_n$ and let A be a $k\varepsilon$-bounded finite automaton accepting L. The length of the longest path in $G(A)$ can be at most $r = (n+1)(k+1) - 1$, since there are n non-ε moves interleaved with at most $k(n+1)$ ε moves in any computation. Therefore, if $G(A)$ can be embedded into a d-dimensional mesh, it can be also embedded into a submesh of $(2r+1) \times (2r+1) \times \ldots \times (2r+1)$ vertices. This submesh contains $(2r+1)^d$ vertices and no more than $\frac{d}{2}(2r+1)^d$ edges. At most six arcs ((p,a,q), (p,b,q), (p,ε,q), (q,a,p), (q,b,p) and (q,ε,p)) can lead through each edge $\{p,q\}$.

We now find an upper bound for the number of distinct finite automata embeddings into a submesh $(2r+1) \times \ldots \times (2r+1)$, assuming some fixed naming of vertices. For example, the set of states can be chosen to be the set of all d-tuples of integers from range $\langle -r, r\rangle$, with initial state $(0,0,\ldots,0)$.

Each of the states can be either final or not and each edge has 2^6 possibilities for the arcs going through. The total number of embeddings is therefore bounded by

$$2^{(2r+1)^d} 2^{6\frac{d}{2}(2r+1)^d} \leq 2^{4d(2r+1)^d}$$

Suppose that every language L from $\mathcal{L}_n$ can be accepted by a $k\varepsilon$-bounded d-mesh finite automaton. This implies that the cardinality of $\mathcal{L}_n$ should be at most $2^{4d(2r+1)^d}$. Since $|\mathcal{L}_n| = 2^{2^n}$, we obtain

$$\begin{aligned}
2^{2^n} &\leq 2^{4d(2r+1)^d} \\
2^n &\leq 4d(2r+1)^d \\
n &\leq \lg(4d) + d\lg(2r+1) = \\
&= \lg(4d) + d\lg[2(k+1)(n+1) - 2 + 1] \leq \\
&\leq \lg(4d) + d[1 + \lg(k+1) + \lg(n+1)]
\end{aligned}$$

Thus

$$n - d\lg(n+1) \leq \lg(4d) + d[1 + \lg(k+1)]$$

The expression on the right hand side of the last inequality is a constant. Clearly, the limit of the left hand side is infinity. It implies that for large n this inequality does not hold. Therefore there must exist an n and a language L in $\mathcal{L}_n$, for which there is no $k\varepsilon$-bounded d-mesh finite automaton. □

Example 4.4 *For a given k we construct a language that cannot be accepted by a 2-mesh $k\varepsilon$-bounded finite automaton.*

Let $L_n = \{ww \mid w \in \{a,b\}^n\}$. By the Myhill-Nerode's Theorem 1.2, there must be a right invariant equivalence class for each word of length n. It implies, that in a finite automaton accepting L there must be at least 2^n states reachable from the initial state in $(n+1)(k+1) - 1$ steps. If we take L_n such that

$$(2(n+1)(k+1) - 1)^2 < 2^n$$

we obtain a regular language that is not recognizable by 2-mesh $k\varepsilon$-bounded finite automata.

5 Conclusions

We have shown that the nondeterministic planar finite automata are equivalent to their ε-free version. Both of them accept exactly the regular languages. Given $L \in \mathcal{R}$, we have obtained a bound on m based on n, where n is the number of classes of equivalence of L (according to the Myhill-Nerode's Theorem 1.2) and m is the number of states of an (ε-free) nondeterministic planar finite automaton accepting L. An upper bound for m and ε-free nondeterministic planar finite automaton obtained from the given construction is $m \leq 3 + \sum_{i=0}^{n-4} 3^i$, for $n \geq 3$. This upper bound holds for every nondeterministic finite automaton.

We leave as an open problem the following conjecture: The lower bound for m is exponential in n for an ε-free nondeterministic planar finite automaton. Similar open problem arises for a nondeterministic planar finite automaton, since the Kleene's construction of regular expressions from a given finite automaton is also exponential in size.

The following problem is also interesting. Given a planar graph G, colour its vertices by colours $1, 2, ..., k$ so that for any sequence $a_1, a_2, ..., a_l$ of these colours there exists a corresponding path in a graph. Try to find an upper bound for k with respect to the number n of vertices of G. If $k = O(\log n)$, our conjecture would be proven.

One more problem in the end: One of the reasons to study embeddings of graphs of automata into a mesh was that all the vertices of a mesh have bounded degree. A more general approach would be to examine the graphs of finite automata that are planar and have a bounded degree. Namely, we are interested in the folowing: For a given n, is there a constant d, such that every regular language over n-letter alphabet can be accepted by a planar finite ε-free (or $k\varepsilon$-bounded) automaton whose graph has maximal degree at most d? We conjecture that the answer is for $n \geq 2$ negative.

References

[1] J. Gruska, Foundations of Computing, International Thompson Computer Press, 1997

[2] S. Ginsburg, Mathematical theory of context-free languages, McGraw-Hill, Inc., 1966, pp.46-55

[3] S. C. Kleene, Representation of events in nerve sets, in C. E. Shannon and J. McCarthy (eds.), "Automata Studies", Princeton University Press, Princeton, N.J., 1956, pp.3-40

[4] J. Myhill, Finite automata and the representation of events, *Wright Air Development Command Tech. Rept.* 57-624, 1957, pp.112-137

[5] A. Nerode, Linear Automaton Transformations, *Proc. Amer. Math. Soc., 9*, 1958, pp.541-544